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SMALL DISTURBANCES OF VERTICAL VELOCITY IN AN ASYMMETRIC ATMOSPHERIC VORTEX

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SMALL DISTURBANCES OF VERTICAL VELOCITY IN AN ASYMMETRIC ATMOSPHERIC VORTEX

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ABSTRACT. The linear theory of disturbances of vertical velocity in an asymmetric vortex is developed. Asymmetry is considered to be a function of the rate of its shift. A closed system of equations of hydrothermodynamics leads to a conventional differential equation for vertical velocity which belongs to the class of degenerate hypergeometric equations. The boundary problem is solved (vertical movements at the center of a vortex are considered zero, and bounded at some finite r = a). Two illustrations are provided with the results of calculations of the vertical velocities at different rates of shift of the cyclone. The configuration of the zones of ascending and descending movements has a spiral appearance and corresponds qualitatively to the distribution of cloud cover in shifting cyclones, observed from artificial Earth satellites. The asymmetry in the distribution of the cloud cover observed in the pictures received from satellites is suggested for use as a sign to determine the direction of shift of cyclones over territories with sparse networks of meteorological stations. 2 Illustrations and 5 Bibliographic entries.

Observations of cloud cover using radar and artificial Earth satellites /35* have revealed quite clearly the spiral shape of the clouds in developed atmospheric vortices.

Papers devoted to an explanation of the mechanism of formation of spiral cloud bands as a rule use a symmetric model of a cyclone [1, 4, 5]. However, television pictures of cloud cover received from satellites as well as radar observations indicate that the spiral cloud bands are quite often arranged somewhat asymmetrically relative to the center of the cyclone. This asymmetry undoubtedly arises as the consequence of the asymmetry of the air flow in the shifting cyclone.

* Numbers in the margin indicate the pagination in the original foreign text.

Unlike previous works, this paper is an attempt to consider in the first approximation the asymmetry of the air flow that arises as the consequence of the shift of the cyclone.

Taking into account the close relationship between cloud cover and the vertical movements, we shall consider in this paper the spatial distribution of the vertical movements, assuming that the structure of the field of vertical movements and the structure of the cloud cover must be closely related to one another. Proceeding on the basis of factual data on the distribution of cloud cover in cyclones, we will consider the disturbances of meteorological elements in the tangential direction to be periodic. Since the spiral shape of the cloud cover assumes periodicity of the disturbances in both the tangential and radial directions, it will be sufficient to obtain the conditions for periodicity of the vertical movements in the radial direction in order to find the conditions that favor the existence of cloud spirals.

The problem has already been solved approximately in this formulation by several authors [1, 5] for the case of a symmetric vortical movement.

We will take as the initial equations, the equations of motion, continuity, heat influx and state, written in cylindrical coordinates. This system of equations will be solved by the method of small disturbances, i.e., we will represent all functions in the original system in the form $f = \bar{f} + f'$, where $\bar{f}$ are the values of the functions in the principal flow and f' are their small disturbances, whose derivatives and squares we shall disregard, thereby limiting ourselves to linear theory.

Let us represent the functions in the original system in the form: /36

$$
\begin{gathered}
u=\overline{u}(r)-u_0 \sin\theta+u'(r)\exp i(mz+n\theta);\\
v=v'(r)\exp i(mz+n\theta);\\
w=w'(r)\exp i(mz+n\theta);\\
p=\overline{p}(r,\ \theta,\ z)+p'(r)\exp i(mz+n\theta);\\
\rho=\overline{\rho}(z)+\rho'(r)\exp i(mz+n\theta);\\
T=\overline{T}(r,\ \theta,\ z)+T'(r)\exp i(mz+n\theta),
\end{gathered}
\tag{1}
$$

where v, u, and w are the radial, tangential and vertical components of the wind speed; ρ is the density, T is the temperature and p is the pressure.

The method of assigning functions characterizing the distributions u, v, w, p, ρ and T in the principal flow (1), assumes a stationary state of the solution. The tangential component of the wind speed is represented in the form of two components, one of which is dependent on the radius and the other on the polar angle. It is easy to see that the term $u_0 \sin\theta$ is the contribution to the tangential component of the wind speed caused by the shift of the cyclone with a constant velocity u_0. We will consider the fluid to be incompressible and $u(r)/r = \omega = \text{const.}$. Performing the simplification assumed in the theory of free convection and disregarding the horizontal temperature gradients in the principal flow, we obtain the system of equations of small disturbances in the form:

$$\left.\begin{aligned}
&\left(\omega - \frac{u_0 \sin\theta}{r}\right) inv' - \left(F - \frac{u_0 \sin\theta}{r}\right) u' = -\frac{1}{\bar{\rho}}\frac{dp'}{dr} \\
&\left(\omega - \frac{u_0 \sin\theta}{r}\right) inu' + \left(F - \frac{u_0 \sin\theta}{r}\right) v' = -\frac{1}{r\bar{\rho}} in p' \\
&\left(\omega - \frac{u_0 \sin\theta}{r}\right) inw' - g\frac{T'}{\bar{T}} = -\frac{1}{\bar{\rho}} imp' \\
&\frac{dv'}{dr} + \frac{v'}{r} + \frac{1}{r} inu' + imw' = 0 \\
&\left(\omega - \frac{u_0 \sin\theta}{r}\right) inT' + (\gamma_a - \gamma) w' = 0
\end{aligned}\right\} \tag{2}$$

where $F = 2\omega + l = \text{const}$, l is the Coriolis parameter, γ_a and γ are the adiabatic and actual temperature gradients.

System (2) can be converted to a conventional second order differential equation for disturbances of the vertical velocity:

$$\begin{aligned}
&\frac{d^2w'}{dr^2} + \left[\frac{1}{r} - \frac{1}{L}\frac{dL}{dr}\left(\frac{\Gamma + 3n^2L^2}{\Gamma - n^2L^2}\right) - \frac{\frac{d}{dr}(R^2 - n^2L^2)}{R^2 - n^2L^2}\right]\frac{dw'}{dr} - \\
&- \left\{\frac{1}{r}\frac{dL}{dr}\left[\frac{(\Gamma - n^2L^2)}{L(\Gamma - n^2L^2)} + \frac{R}{L^2} + \frac{2Rn^2}{(\Gamma - n^2L^2)} - \frac{R(\Gamma + n^2L^2)}{L^2(\Gamma - n^2L^2)}\right] - \right. \\
&\quad - \left(\frac{dL}{dr}\right)^2 - \frac{1}{rL}\frac{dR}{dr} + \frac{(\Gamma + n^2L^2)}{(\Gamma - n^2p^2)}\frac{1}{L}\frac{d^2L}{dr^2} -
\end{aligned} \tag{3}$$

$$-\frac{\frac{d}{dr}(R^2-n^2L^2)}{R^2-n^2L^2}\left[\frac{(\Gamma+n^2L^2)}{(\Gamma-n^2L^2)}\frac{1}{L}\frac{dL}{dr}-\frac{R}{rL}\right]+\frac{n^2}{r^2}+$$
$$+\frac{m^2(R^2-n^2L^2)}{(\Gamma-n^2L^2)}\Bigg\}w'=0, \tag{3}$$

where

/37

$$L=F-\frac{u_0\sin\theta}{r};\quad R=\omega-\frac{u_0\sin\theta}{r};\quad \Gamma=\frac{g(\gamma_a-\gamma)}{\overline{T}}. \tag{4}$$

We then estimate the order of the terms entering into the coefficients of (3), keeping in mind the characteristic dimensions of cyclones (on the order of hundreds of kilometers). Then

$$O\left[\frac{1}{r}\right]\sim 10^{-5}\ \text{м}^{-1};\quad O[R]\sim 10^{-5}\ \text{sec}^{-1};$$
$$O[L]\sim 10^{-5}\ \text{sec}^{-1};\quad O[L]'\sim 10^{-5}\ \text{sec}^{-2}. \tag{5}$$

By substituting the characteristic values of the parameters from (5) into (3), it is easy to see that the additives containing the derivatives can be disregarded for r > 100. Hence, for a large part of the cyclone (with the exception of a small region adjacent to its center) the nature of the small disturbances can be described rather accurately by the equation

$$\frac{d^2w'}{dr^2}+\frac{1}{r}\frac{dw'}{dr}-\left[\frac{n^2}{r^2}+\frac{m^2(R^2-n^2L^2)}{(\Gamma-n^2L^2)}\right]w'=0. \tag{6}$$

Performance of the calculations for the symmetric model [1] has shown that under actual conditions in the atmosphere the inequality $\Gamma\gg n^2L^2$ is nearly always fulfilled. The exception is the case of adiabatic stratification of the atmosphere. Taking into account the fact that the vertical temperature gradients in a rather thick layer of the atmosphere (approximately 1 km and more) rarely exceed the value 0.8°/100 m, we can use the inequality given above and write (6) as follows:

$$\frac{d^2w'}{dr^2}+\frac{1}{r}\frac{dw'}{dr}-\left[\frac{n^2}{r^2}+\frac{m^2(R^2-n^2L^2)}{\Gamma}\right]w'=0. \tag{7}$$

This is a degenerate hypergeometric equation and its solution is found with the aid of degenerate hypergeometric functions. It should be noted that in the event of a symmetric state of the principal flow, (7) is converted to the Bessel function discussed earlier by the author [1].

We then regroup the additives in the coefficient with the free term in (7), rewriting it in the form

$$\frac{d^2w'}{dr^2}+\frac{1}{r}\frac{dw'}{dr}-\left[\alpha^2-\frac{\beta}{r}+\frac{\lambda^2}{r^2}\right]w'=0, \tag{8}$$

where

$$\alpha^2=\frac{m^2(F^2-n^2\omega^2)}{\Gamma};\quad \beta=2m^2u_0\sin\theta\frac{(F-n^2\omega)}{\Gamma}$$
$$\lambda^2=n^2+\frac{m^2}{\Gamma}u_0^2\sin^2\theta(1-n^2). \tag{9}$$

After substituting $w'=\exp\left[-\frac{1}{2}\int\frac{1}{r}dr\right]\tilde{w}$, this equation assumes the form

$$\frac{d^2\tilde{w}}{dr^2}+\left[-\alpha^2+\frac{\beta}{r}+\frac{\frac{1}{4}-\lambda^2}{r^2}\right]\tilde{w}=0. \tag{10}$$

Assuming $\lambda=l+\frac{1}{2}$, $r_1=\frac{1}{2}\beta r$. we will have /38

$$\frac{d^2\tilde{w}}{dr_1^2}+\left[-\frac{4\alpha^2}{\beta^2}+\frac{2}{r_1}-\frac{l(l+1)}{r_1^2}\right]\tilde{w}=0. \tag{11}$$

Finally, carrying out the substitution

$$r_2=(8r_1)^{\frac{1}{2}},\quad \tilde{w}=\frac{1}{2}r_2\tilde{w}, \tag{12}$$

we obtain (11) in the form

$$\nabla l\tilde{w}^{(l,\beta)}-\frac{1}{\beta^2}\left(\sqrt{\frac{\alpha}{2}}r_2\right)^4\tilde{w}^{(l,\beta)}=0, \tag{13}$$

where

$$\nabla l=r_2^2\frac{d^2}{dr_2^2}+r^2\frac{d}{dr_2}+r_2^2-(2l+1)^2$$

— the Bessel operator with subscript $2l+1$.

The general solution of (13) can be written (see [3]) in the form

$$\tilde{w}(r_2) = CJ^3_{2l+1}(r_2) + DN^3_{2l+1}(r_2), \tag{14}$$

where C and D are arbitrary constants and J^3_{2l+1} and N^3_{2l+1} are hypergeometric functions. The general solution of (14) is a linear combination of two independent solutions satisfying different boundary conditions.

Cloudy and cloudless spirals alternately converge on the center of the cyclone. On this basis, it is advantageous in the solution of (14) to consider the disturbances in vertical velocity at the center of the cyclone to be equal to zero. We will consider that the disturbances of the vertical velocity at the periphery of the cyclone are known:

$$w'|_{r=0} = 0; \quad w'|_{r=a} = w'_0. \tag{15}$$

It is easy to show that in this case the following boundary conditions are fulfilled:

$$\tilde{w}|_{r_2=0} = 0; \quad \tilde{w}|_{r_2=0} = \tilde{w}_0.$$

We can show [3] that the only solution of (14) which satisfies the Conditions (15) is

$$\tilde{w}(r_2) = CJ^3_{2l+1}(r_2). \tag{16}$$

The second independent solution has the pole at the origin of the coordinates and does not satisfy the selected boundary conditions. Considering Relationships (15) and shifting to the original variables and functions, we obtain the final solution

$$w'(r) = \frac{w_0'\sqrt{a}}{\sqrt{r}} M_{\beta,\lambda}(2\alpha r) \tag{17}$$

where $M_{\beta,\lambda}(2\alpha r)$ is the Whittaker function.

Substituting (17) in the third relationship in System (1), we obtain the general expression for disturbances of the vertical velocity:

$$w'(r,\ \theta,\ z) = \exp i(mz + n\theta)\frac{w_0'\sqrt{a}}{\sqrt{r}} M_{\beta,\lambda}(2\alpha r). \tag{18}$$

Inasmuch as the disturbances of vertical velocity that appear when clouds develop in the wave troughs are projected in the horizontal plane in television pictures of cloud cover, it is most interesting to view them relative to the coordinates r and θ, considering that the disturbances are located at some fixed height $\bar{z}$. Performing for the sake of simplicity the calculations $\bar{z} = L_z/4$, where $L_z = 2\pi/m$ is the vertical wave length and selecting the real part in (18), we will have /39

$$Re\,w'(r,\ \theta) = -\frac{w_0'\sqrt{a}}{\sqrt{r}} M_{\beta,\lambda}(2\alpha r)\sin n\theta. \tag{19}$$

The calculation of the functions $M_{\beta,\lambda}(2\alpha r)$ is performed with the aid of a series [2]

$$M_{\beta,\lambda}(2\alpha r) = (2\alpha r)^{\frac{1}{2}+\lambda} e^{-\alpha r}\left\{1 + \frac{\frac{1}{2}+\lambda-\beta}{1!(2\lambda+1)}2\alpha r + \right.$$
$$\left. + \frac{\left(\frac{1}{2}+\lambda-\beta\right)\left(\frac{3}{2}+\lambda-\beta\right)}{2!(2\lambda+1)(2\lambda+2)}(2\alpha r)^2 + \ldots\right\}. \quad (20$$

The calculations using (19) and (20) are cumbersome, so that when the numerical experiment was performed some further simplifications were made which are completely admissible in considering the real atmospheric processes.

We will estimate the order of the values β and λ entering into Series (20). Assuming L_z = 8 km, u_0 = 10 m/sec, $\bar{T} = 270°$, $O[F] = O[\omega] = 10^{-5}$ sec^{-1}, $O[\gamma_a - \gamma] = 10^{-3}$ deg/m, we will have

$$O[\beta] = 10^{-6}\ \text{м}^{-1}, \quad O]\lambda] = 1\ \text{м}^{-1},$$

i.e., the consideration of β makes a negligible contribution to the value of the vertical velocity. Therefore, (8) can be replaced with a sufficient degree of accuracy by the Bessel Equation (21)

$$\frac{d^2w'}{dr^2} + \frac{1}{r}\frac{dw'}{dr} + \left[\alpha_1^2 - \frac{\lambda^2}{r^2}\right] w' = 0, \tag{21}$$

where

$$\alpha_1^2 = \frac{m^2(n^2\omega^2 - F^2)}{\Gamma}. \tag{22}$$

Equation (21) and the boundary Conditions (15) are satisfied by the solution

$$w' = CJ_\lambda(\alpha_1 r), \tag{23}$$

and the solution analogous to (19) assumes the form

$$Re\,w'(r, \theta) = -\frac{w_0'}{J(\alpha_1 a)} J_\lambda(\alpha_1 r) \sin n\theta \tag{24}$$

Thus, the nature of the distribution of vertical velocities depends on the parameters α_1 and λ. For realization of a periodic motion in the radial direction (and this is precisely what is required for the formation of spiral /40 cloud structure), the following inequality must be satisfied:

$$\alpha_1 = m\sqrt{\frac{n^2\omega^2 - F^2}{\Gamma}} > 0. \tag{25}$$

The parameter λ determines the order of the Bessel function. Proceeding from (9), we formally have two roots:

$$\lambda_1 = n - \frac{m}{\sqrt{\Gamma}} u_0 \sin \theta \sqrt{n^2 - 1},$$
$$\lambda_2 = n + \frac{m}{\sqrt{\Gamma}} u_0 \sin \theta \sqrt{n^2 - 1},$$

The physical conditions of the problem are satisfied only by the root λ_1, since it is only in this case that there is a correct consideration of the contribution of translation to the tangential component (the tangential component must increase to the right of the direction of shift of the cyclone and must increase to the left).

We used (24) in calculating the vertical velocity in the region of cyclones of different intensity,moving at different velocities. Without dwelling on the effects of stratification and angular velocity (ω), let us see how the shift if cyclones affects asymmetry in the distribution of vertical movements.

Figures 1 and 2 show the distribution w' at cyclone shift rates u_0 equal to 7 and 15 m/sec, respectively. It is easy to see in Figure 1 the asymmetry of the distribution of vertical movements. As the rate of shift of the vortex increases, asymmetry is observed with increasing clarity, as we can conclude by comparing Figures 1 and 2. As a the angular velocity increases, the asymmetry becomes slightly blurred, but at a cyclone shift rate in excess of 10 m/sec it is quite clearly evident. /41

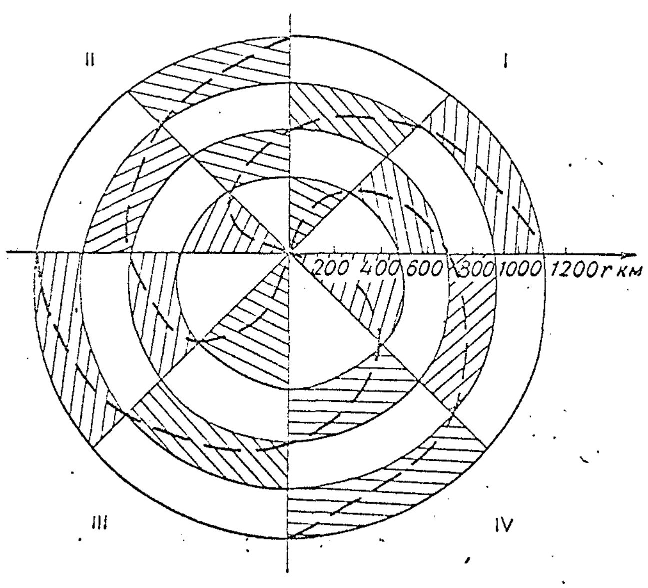

Figure 1. Distribution of small disturbances of vertical velocity at u_0 = 7 m/sec. Areas with rising movements are shaded in the figure. The arrow indicates the direction of shift of the vortex.

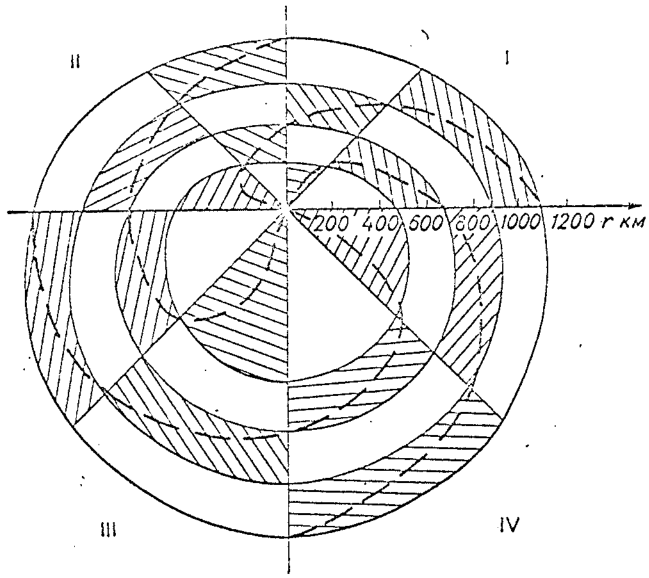

Figure 2. Distribution of small disturbances of vertical velocity at $u_0 = 15$ m/sec.

The asymmetry in the distribution of vertical currents (and therefore, in the cloud cover) is obviously the result not only of shifting of the cyclone, but also of thermal inhomogeneity of the cyclone itself and other factors. Nevertheless, the asymmetry in the distribution of the cloud cover which is often observed in pictures received from artificial Earth satellites can be used as a guide to determine the direction of shift of cyclones over a territory with a sparse network of meteorological stations.

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